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The fundamental algebraic properties of split quasi-octonions

Split kuazi-oktonyonların temel cebirsel özellikleri

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ABSTRACT

The fundamental properties of split quasi-octonion algebra, O'q, and definitions of fundamental operations such as scalar and vector parts, conjugate, norm, and polar form are presented. We explain the Cayley-Dickson construction of split quasi-octonion algebra, in particular we provide table of the octonion multiplication.

Keywords: Alternativity, Cayley-Dickson construction, Split quasi-octonion, Trigonometric form

ÖZET

Split kuaz-oktonyonların temel cebirsel özellikleri ve bazı temel operasyonlar tanımları, örneğin, skalar ve vektör parçaları, eşlenik, norm ve polar formu sunulmuştur. Split kuazi-oktonyonlar cebirin üzerinde Cayley-Dickson yapısını açıkladık, ve oktoniyon çarpım tablosunu temin ettik.

Anahtar sözcükler: *Alternativite, Cayley-Dickson yapısı, Split kuazi-octonion, Trigonometrik form*

Introduction

The Octonion, or the Cayley algebra O is an 8-dimensional non-associative algebra, which is defined by J.T. Graves and A. Cayley independently separated. Since octonions share with complex numbers and quaternions have many attractive mathematical properties, one might except that they would be equally useful. As a vector space, the octonions are

$$O=\left\{a_0+\sum_{i=1}^{7}a_ie_i\colon a_0,a_1,...a_7\in R\right\}$$

In our previous work, we investigated basic algebraic properties of real, split, complex, semi, and quasi octonions algebra. In following studies, here we study fundamental properties of split quasi-octonions, which is called split $\frac{1}{4}$-octonions in [9]. We review the generalized octonions algebra, and show that if put $\alpha=-1,\beta=\gamma=0$ is obtained split quasi-octonions algebra. Like real octonions, split semi-octonions form a non-associative algebra, but unlike real octonions, they are not division algebra. By Cayley-Dickson construction, e4 and H generates O_q' as an algebra. We express any split quasi-octonions in trigonometric form similar to octonions and quaternions. In addition, we prove De Moivre's theorem and Euler's formula for these octonions.

1. Generalized Octonions Algebra

In this section, we give a brief summary of the generalized octonions. For detailed information about these octonions, we refer the reader to [1].

Definition 2.1. A generalized octonion x is defined as

$$x=a_0a_0+a_1a_1+a_2a_2+a_3a_3+a_4a_4+a_5a_5+a_6a_6+a_7a_7,$$

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where $a_0 - a_7$ are real numbers and $e_i, (0 \le i \le 7)$ are octonionic units satisfying the equalities that are given in the following table;

.	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	$-\alpha$	e_3	$-\alpha e_2$	e_5	$-\alpha e_4$	e_7	αe_6
e_2	e_3	$-\beta$	βe_1	e_6	e_7	$-\beta e_4$	$-\beta e_5$
e_3	αe_2	$-\beta e_1$	$-\alpha\beta$	e_7	$-\alpha e_6$	βe_5	$-\alpha\beta e_4$
e_4	e_5	e_6	e_7	$-\gamma$	γe_1	γe_2	γe_3
e_5	αe_4	$-e_7$	αe_6	$-\gamma e_1$	$-\alpha\gamma$	$-\gamma e_3$	$\alpha\gamma e_2$
e_6	e_7	βe_4	$-\beta e_5$	$-\gamma e_2$	γe_3	$-\beta\gamma$	$-\beta\gamma e_1$
e_7	$-\alpha e_6$	βe_5	$\alpha\beta e_4$	$-\gamma e_3$	$-\alpha\gamma e_2$	$\beta\gamma e_1$	$-\alpha\beta\gamma$

Special Cases:

1. If $\alpha = \beta = \gamma = 1$, is considered, then $O(\alpha,\beta,\gamma)$ is the algebra of real octonions O [5].
2. If $\alpha = \beta = 1, \gamma = -1$, is considered, then is the algebra of split octonions (Psoudo-octonions) O' [4].
3. If $\alpha = \beta = 1, \gamma = 0$, is considered, then $O(\alpha,\beta,\gamma)$ is the algebra of semi-octonions O_S [3].
4. If $\alpha = \beta = -1, \gamma = 0$, is considered, then $O(\alpha,\beta,\gamma)$ is the algebra of split semi-octonions O'_S [5].
5. If $\alpha = 1, \beta = \gamma = 0$, is considered, then $O(\alpha,\beta,\gamma)$ is the algebra of quasi-octonions O_q [6].
6. If $\alpha = -1, \beta = \gamma = 0$, is considered, then $O(\alpha,\beta,\gamma)$ is the algebra of split quasi-octonions O'_q.
7. If $\alpha = \beta = \gamma = 0$, is considered, then $O(\alpha,\beta,\gamma)$ is the algebra of para-octonions O_p [7].

The generalized octonions algebra, $O(\alpha,\beta,\gamma)$, is a non-commutative, non-associative, alternative, flexible and power-associative.

2. Split Quasi-Octonions Algebra

Definition 3.1. A split quasi-octonion x is expressed as a set of eight real numbers

$$x = (x_0, x_1, ..., x_7) = x_0 e_0 + \sum_{i=1}^{7} x_i e_i,$$

where $x_0 - x_7$ are real numbers. The multiplication rules among the basis elements of octonions $e_i (0 \le i \le 7)$ can be expressed in the form:

$e_1^2 = 1,$ $e_k^2 = 0,\ 2 \le k \le 7,$
$e_1 e_2 = e_3 = -e_2 e_1,$ $e_2 e_4 = e_6 = -e_4 e_2$ $e_1 e_3 = e_2 = -e_3 e_1,$
$e_1 e_4 = e_5 = -e_4 e_1,$ $e_2 e_5 = e_7 = -e_5 e_2$ $e_1 e_5 = e_4 = -e_5 e_1$
$e_1 e_6 = -e_7 = -e_6 e_1,$ $e_3 e_4 = e_7 = -e_4 e_3$ $e_3 e_5 = e_6 = -e_5 e_3$

The above multiplication rules are given in the following Table;

.	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	1	e_3	e_2	e_5	e_4	e_7	$-e_6$
e_2	e_3	0	0	e_6	e_7	0	0
e_3	$-e_2$	0	0	e_7	e_6	0	0
e_4	e_5	e_6	e_7	0	0	0	0
e_5	$-e_4$	$-e_7$	$-e_6$	0	0	0	0
e_6	e_7	0	0	0	0	0	0
e_7	e_6	0	0	0	0	0	0

By using the Cayley-Dikson construction, a split quasi-octonion *x* can also be written as

$$x = (a_0 e_0 + a_1 e_1 + a_2 e_2 + a_3 e_3) + (a_4 + a_5 e_1 + a_6 e_2 + a_7 e_3) e_4 = q + q' l,$$

where $l^2 = 0$ and q, q' are split semi-quaternions [2], *i.e.*

$$q, q' \in H_{SS}^{O} = \left\{ q = a_0 + a_1 e_1 + a_2 e_2 \,\middle|\, e_1^2 = 1, e_2^2 = e_3^2 = 0, a_i \in R \right\}$$

This construction lets us view the split quasi-octonion as a two dimensional vector space over split semi-quaternions quaternions. Therefore, $O'_q = H_{SS}^{O} \oplus H_{SS}^{O} l.$

A split quasi-octonion x can be decomposed in terms of its scalar (S_x) and vector $(\vec{V}_x)$ parts as

$$S_x = a_0,\ \vec{V}_x = a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5 + a_6 e_6 + a_7 e_7.$$

For two split semi-octonions $x = \sum_{i=0}^{7} a_1 e_1$ and $w = \sum_{i=0}^{7} b_1 e_1$ the summation and substraction processes are given as $x \pm w = \sum_{i=0}^{7} (a_i \pm b_i) e_i$

The product of two split semi-octonions $x = S_x + \vec{V}_x, w = S_w + \vec{V}_w$ is expressed as

$$xw = S_x S_w - \langle \vec{V}_x, \vec{V}_w \rangle + S_x \vec{V}_w + S_w \vec{V}_x + \vec{V}_x \mathrm{x} \vec{V}_w$$

This product can be described by a matrix-vector product as

$$x.w = \begin{pmatrix} a_0 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & b_0 \\ a_1 & a_0 & 0 & 0 & 0 & 0 & 0 & 0 & b_1 \\ a_2 & -a_3 & a_0 & a_1 & 0 & 0 & 0 & 0 & b_2 \\ a_3 & -a_2 & a_1 & a_0 & 0 & 0 & 0 & 0 & b_3 \\ a_4 & -a_5 & 0 & 0 & a_0 & a_1 & 0 & 0 & b_4 \\ a_5 & -a_4 & 0 & 0 & a_1 & a_0 & 0 & 0 & b_5 \\ a_6 & a_7 & -a_4 & -a_5 & a_2 & a_3 & a_0 & a_1 & b_6 \\ a_7 & a_6 & -a_5 & -a_4 & a_3 & a_2 & -a_1 & -a_0 & b_7 \end{pmatrix}$$

Split semi-octonions multiplication is not associative, since

$$e_1(e_2e_4) = e_1e_6 = -e_7,$$
$$(e_1e_2)e_4 = e_3e_4 = e_7.$$

But it has the property of *alternativity*, that is, any two elements in it generate an associative subalgebra isomorphic to $\mathrm{R}, \mathbb{C}^0, \mathbb{C}^1, \mathrm{H_S}, \mathrm{H}^0$.

e_0 and $e_i (2 \le i \le 7)$ generate a subalgebra isomorphic to $\mathbb{C}^0$ (dual numbers),

e_0 and e_1 generate a subalgebra isomorphic to $\mathbb{C}^1$ (double (or split complex) numbers),

Subalgebra with bases $e_0, e_1, e_i, e_j (2 \le i, j \le 7)$ is isomorphic to split semi-quaternions algebera H_S ([2])

Subalgebra with bases $e_0, e_1, e_j, e_k (2 \le i, j, k \le 7)$ is isomorphic to quasi-quaternions algebra H^0.

3. Some Properties of Split Quasi-Octonions

1) The *conjugate* of split quasi-octonion $x = \sum_{i=0}^{7} a_i e_i = S_x + \overset{r}{V}$ is

$$\bar{x} = a_0 e_0 - \sum_{i=1}^{7} a_i e_i = S_x - \overset{r}{V}_x.$$

Conjugate of product of two split quasi-octonions and its own are described as

$$\overline{xy} = \bar{y}\bar{x}, \bar{\bar{x}} = x$$

It is clear the scalar and vector parts of x is denoted by $S_x = \dfrac{x+\bar{x}}{2}$ and $\overset{r}{V}_x = \dfrac{x-\bar{x}}{2}$.

2) The *norm* of x is

$$N_x = x\bar{x} = \bar{x}x = |x|^2 = a_0^2 - a_1^2.$$

It satisfies the following property

$$N_{xy} = N_x N_y = N_y N_x$$

The modulus $|x|$ of a split quasi-octonion *x*, like the modulus of a split quaternion, or split octonion, can be real or imaginary and can be equal to 0 for $x \neq 0$.

A split quasi-octonion *x* is called quasi-spacelike, quasi-timelike or quasi-lightlike(null), if $N_x < 0$, $N_x > 0$ or $N_x = 0$, respectively.

If $N_x = 1$, then x is called a unit split quasi-octonion. We will use O'_{q_1} to denote the set of unit split quasi-octonions.

3) The *inverse* of x with $N_x \neq 0$, is

$$x^{-1} = \frac{1}{N_x}\bar{x}.$$

4) The *trace* of element x is defined as $t(x) = x + \bar{x}$

For every $x \in O'_q$, we have $(x+\bar{x})x = x^2 + \bar{x}x = x^2 + N_x.1$, then, $x^2 - t(x)x + N_x = 0$, therefore, the split quasi-octonions algebra is quadratic.

The split quasi-octonions algebra is not division algebra, because for every nonzero $x \in O'_q$ the relation $N_x = 0$, implies $x \neq 0$.

Example 4.1. Consider the split quasi-octonions

$$x_1 = 2 + (1, -1, 2, -2, 0, 1, 1)$$

$x_2 = -1 + (2, -1, 1, -2, 0, 1, 1)$ and

$$x_3 = \frac{-1}{2} + (\frac{1}{2}, -1, \sqrt{2}, -2, 2, 1, 1);$$

1. The vector parts of x_1, x_2 are

$\overset{r}{V}_{x_1} = (1, -1, 2, -2, 0, 1, 1), \overset{r}{V}_{x_2} = (2, -1, 1, -2, 0, 1, 1),$

2. The conjugates of x_1, x_2 are

$\bar{x}_1 = 2 - (1, -1, 2 -2, 0, 1, 1), \bar{x}_2 = -1 - (2, -1, 1 -2, 0, 1, 1).$

3. The norms are given by

$$N_{x_1} = 3, N_{x_2} = -3, N_{x_3} = 0.$$

4. The inverses are

$$x_1^{-1} = \frac{1}{N_{x_1}}\bar{x}_1 = \frac{2}{3} - \frac{1}{3}(1, -1, 2, -2, 0, 1, 1),$$

$x_2^{-1} = \dfrac{1}{3} + \dfrac{1}{3}(2, -1, 1, -2, 0, 1, 1)$, and x_3 not invertible.

5. One can realize the following operations

$$x_1 + x_2 = 0 + (2,-2,3,-4,0,2,2)$$
$$x_1 - x_2 = 2 + (0,0,1,0,0,0)$$
$$x_1 x_2 = -1 + (1,-1,-1,-1,0,2,-2)$$
$$x_2 x_1 = -1 + (1,1,-1,1,1,2,-)$$
$$N_{x_1 x_2} = N_{x_1} N_{x_2} = N_{x_2 x_1} = 0.$$

Theorem 4.1. The set $O^1_{S_1}$ of unit split semi-octonions is a subgroup of the group $O^1_{S_0}$ where $O^O_S = O_S - [0 - \overset{1}{0}]$.

Proof: Let $x, y \in O'_{S_1}$. We have $N_{xy} = 1$ *i.e.* $xy \in O'_S$ and thus the first subgroup requirement is satisfied. Also, by the property

$$N_x = N_{\bar{x}} = N_{x^{-1}} = 1,$$

the second subgroup requirement $x^{-1} \in O'_{S_1}$.

4. Trigonometric Form and De Moivre's Theorem

Trigonometric(polar) form of the nonzero split quasi-octonion $x = \sum_{i=0}^{7} a_i e_i$ is as follows:

i) Every quasi-spacelike octonion x can be written in the form $x = \sqrt{|N_x|}(\sinh\lambda + \vec{w}\cosh\lambda)$

where

$$\sinh\lambda = \frac{a_0}{\sqrt{|N_x|}},\ \cosh\lambda = \frac{\sqrt{a_1^2}}{\sqrt{|N_x|}} = \frac{|a_1|}{\sqrt{|N_x|}}$$

the unit octonion vector $\vec{w}$ is given by

$$\vec{w} = (w_1, w_2, ..., w_7) = \frac{1}{\sqrt{a_1^2}}(a_1, a_2, ..., a_7).$$

Since $\vec{w}^2 = 1$; we have a natural generalization of Euler's formula for unit split quasi-octonion

$$e^{\vec{w}\lambda} = 1 + \vec{w}\lambda + \frac{(\vec{w}\lambda)^2}{2!} + \frac{(\vec{w}\lambda)^3}{3!} + \frac{(\vec{w}\lambda)^4}{4!} + \frac{(\vec{w}\lambda)^5}{5!} + ...$$
$$= (1 + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + ...) + (\vec{w}\lambda + \frac{(\vec{w}\lambda)^3}{3!} + \frac{(\vec{w}\lambda)^5}{5!} + ...)$$
$$= \cosh\lambda + \vec{w}\sinh\lambda$$

Example 5.1. The trigonometric forms of the split quasi-octonions [1]

$x_1 = 1 + (2,-1,0,1,1,1,-1)$ is

$$x_1 = \sqrt{3}[\sinh \ln\sqrt{3} + \vec{w}_1 \cosh \ln\sqrt{3}],$$

$x_2 = 1 + (\sqrt{2},-1,0,1,-1,2,1)$ is

$$x_2 = \sinh \ln(1+\sqrt{2}) + \vec{w}_2 \cosh \ln(1+\sqrt{2})$$

where

$$\vec{w}_1 = \frac{1}{2}(2,-1,0,1,1,1,-1)$$
$$\vec{w}_2 = \frac{1}{\sqrt{2}}(\sqrt{2},-1,0,1,-1,2,1)\ and\ N_{\vec{w}_1} = N_{\vec{w}_2} = -1.$$

ii) Every quasi-timelike octonion x can be written in the form

$$x = \sqrt{N_x}(\cosh\theta + \vec{u}\sin\theta)$$

where

$$\cosh\theta = \frac{a_0}{\sqrt{N_x}}, \sinh\theta = \frac{\sqrt{a_1^2}}{\sqrt{N_x}} = \frac{|a_1|}{\sqrt{N_x}},$$

the unit octonion vector $\vec{u}$ $(N_{\vec{u}} = 1)$ is given by

$$\vec{u} = (u_1, u_2, ..., u_7) = \frac{1}{\sqrt{a_1^2}}(a_1, a_2, ..., a_7).$$

Example 5.2. The polar forms of the split quasi-octonions

$x_1 = \frac{1}{\sqrt{2}} + (-\frac{1}{2},1,-1,1,2,1,-2)$ is

$$x_1 = \frac{1}{2}[\cosh \ln(1+\sqrt{2}) + \vec{u}_1 \sinh \ln(1+\sqrt{2})],$$

$x_2 = \sqrt{3} + (-\sqrt{2},0,\sqrt{2},-1,1,-1,1)$ is

$$x_2 = \cosh \ln(\sqrt{2}+\sqrt{3}) + \vec{u}_2 \sinh \ln(\sqrt{2}+\sqrt{3}),$$

where

$$\vec{u}_1 = 2(-\frac{1}{2},1,-1,1,2,1,-2), \vec{u}_2 = \frac{1}{\sqrt{2}}(-\sqrt{2},0,\sqrt{2},-1,1,-1,1)$$

iii) Every null octonion x can be written in the form

$$x = 1 + \vec{\varepsilon}$$

where $\vec{\varepsilon}$ is a null vector ($N_{\vec{\varepsilon}} = -1$).

Example 5.3. The polar form of the split quasi-octonions $x = 1 + (1,0,-1,1,1,-1,-2)$ is $x = 1 + \vec{\varepsilon}$

where $\vec{\varepsilon} = (1,0,-1,1,1,-1,-2)$.

Theorem 5.1. (De Moivre's formula)Let $x = \sqrt{|N_x|}(\sinh\lambda + \vec{w}\cosh n\lambda)$ be a quasi-spacelike octonion. We have

$$x^n = (\sqrt{|N_x|})^n(\sinh\lambda + \vec{w}\cosh n\lambda) \text{ for } n \text{ odd}$$

and

$$x^n = (\sqrt{|N_x|})^n(\cosh n\lambda + \vec{w}\sinh n\lambda) \text{ for } n \text{ even.}$$

1 The inverse hyperbolic sine and cosine are defined $\sinh^{-1}x = Ln(x + \sqrt{x^2+1})$ and $\cosh^{-1}x = Ln(x + \sqrt{x^2-1}), (x > 1)$

Proof: The proof is easily followed by induction on *n*.

Example 5.4. Let $x = 1 + (-\sqrt{2}, -1, 0, 1, 2, 2, -1)$. Find x^{26} and x^{43}

Solution: First write x in trigonometry form:

$$x = \sinh \ln(1+\sqrt{2}) + \overset{r}{w} \cosh \ln(1+\sqrt{2})$$

$$x^{25} = \cosh 26[\ln(\sqrt{2}+1)] + \overset{r}{w} \sinh 26[\ln(\sqrt{2}+1)].$$

$$x^{43} = \sinh 43[\ln(\sqrt{2}+1)] + \overset{r}{w} \cosh 43[\ln(\sqrt{2}+1)].$$

Theorem 5.2. (De Moivre's formula) Let $x = \sqrt{N_x}(\cosh\varphi + \overset{r}{v}\sinh\varphi)$ be a quasi-timelike octonion. Then for any integer

$$x^n = (\sqrt{N_x})^n(\cosh n\varphi + \overset{r}{u}\sinh n\varphi)$$

Proof: The proof is easily followed by induction on *n*.

Theorem 5.3. (De Moivre's formula)If $x = \sum_{i=0}^{7} a_i e_i = 1 + \overset{r}{\varepsilon}$ be a null octonion. Then for any integer

$$x^n = 1 + n\overset{r}{\varepsilon}.$$

5. The roots of a Split Quasi-Octonion

Theorem6.1. Let $x = \sqrt{|N_x|}(\sinh\lambda + \overset{r}{w}\cosh\lambda)$ be a quasi-spacelike octonion. The equation $a^n = x$ has only one root and this is

$$a = \sqrt[2n]{|N_x|}(\sinh\frac{\lambda}{n} + \overset{r}{w}\cosh\frac{\lambda}{n})$$

Theorem 6.2. Let $x = \sqrt{N_x}(\cosh\lambda + \overset{r}{v}\sinh\lambda)$ be a quasi-timelike octonion. The equation $a^n = x$ has only one root and this is

$$a = \sqrt[2n]{N_x}(\cosh\frac{\lambda}{n} + \overset{r}{v}\sinh\frac{\lambda}{n})$$

Proof: We assume that $a = M(\cosh\lambda + \overset{1}{v}\sinh\lambda)$ is a root of the equation $a^n = x$, since the vector parts of x and a are the same. From Theorem 5.2, we have

$$a^n = M^n(\cosh n\lambda + \overset{r}{v}\sinh n\lambda)$$

Now, we find

$$M = \sqrt{N_x}, \quad \cosh\varphi = \cosh n\lambda, \qquad \sinh\varphi = \sinh n\lambda.$$

So, $a = \sqrt[n]{N_x}(\cosh\frac{\varphi}{n} + \overset{r}{v}\sinh\frac{\varphi}{n})$ is a root of equation $a^n = x$. If we suppose that there are two roots satisfying the equality, we obtain that these roots must be equal to each other.

Example 6.1. Let $x = \sqrt{3} + (\sqrt{2}, 1, 2-1, 1, -1, 1)$. Find $x^{\frac{1}{4}}$

Solution: First we write *x* in polar form:

$$x = \cosh\ln(\sqrt{2}+\sqrt{3}) + \overset{r}{u}_2 \sinh\ln(\sqrt{2}+\sqrt{3}),$$

Then,

$$x^{1/4} = \cosh\frac{\ln(\sqrt{2}+\sqrt{3})}{4} + \overset{r}{u}_2\sinh\frac{\ln(\sqrt{2}+\sqrt{3})}{4},$$

Conclusion

In this paper, we defined and gave some of algebraic properties of split quasi-octonions and investigated the De Moivre's formulas for these octonions. We gave some examples for more clarification.

We hope that this work will contribute to the study of physics and other sciences.

Futher Work

We will give a complete investigation to real matrix representations of split quasi-octonions, and consider a relation between the powers of these matrices.

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